On the conventions

Antonino D'Anna

1 General conventions

Here we fix some general convention that will be intended thoughout the codebase and generally the data analysis.

1.1 γ-matrices

We use the euclidean γ-matrix defined as

$$\gamma_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{bmatrix} \tag{1}$$

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \tag{2}$$

in particular we have

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_5, \gamma_\mu\} = 0 \tag{3}$$

$$\gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_5^\dagger = \gamma_5 \tag{4}$$

$$\gamma_5^2 = \gamma_\mu^2 = \mathbb{1} \tag{5}$$

1.2 Bilinear Operator

We define a general operator bilinear as

$$O_\Gamma^{ij}(x) = q^i(x)\Gamma q^j(x) \tag{6}$$

and the complete list of operator bilinear is

$$\begin{aligned} P^{ij}(x) =& \overline{q}^i(x)\gamma_5 q^j(x) & (7)\\ A^{ij}_\mu(x) =& \overline{q}^i(x)\gamma_\mu\gamma_5 q^j(x) & (8)\\ V^{ij}_\mu(x) =& \overline{q}^i(x)\gamma_\mu q^j(x) & (9)\\ T^{ij}_{\mu\nu}(x) =& \overline{q}^i(x)\frac{1}{2}[\gamma_\mu,\gamma_\nu]q^j(x) & (10)\end{aligned}$$

We also define $\overline{O}^{ij}_\Gamma$ as

$$\overline{O}^{ij}_\Gamma = \overline{q}^j(x)\gamma_0\Gamma^\dagger\gamma_0 q^i(x) = \eta_\Gamma O^{ji}_\Gamma(x) \tag{11}$$

where η_Γ is uniquely determined by:

$$\gamma_0\Gamma^\dagger\gamma_0 = \eta_\Gamma\Gamma \tag{12}$$

and for the bilinear in eqs (7-10), we have

$$\begin{aligned} \overline{P}^{ij}(x) =& -P^{ji}(x) & (13)\\ \overline{A}^{ij}_\mu(x) =& (-1)^{\delta_{\mu,0}-1}A^{ji}_\mu(x) & (14)\\ \overline{V}^{ij}_\mu(x) =& (-1)^{\delta_{\mu,0}-1}V^{ji}_\mu(x) & (15)\\ \overline{T}^{ij}_{\mu\nu}(x) =& (-1)^{\delta_{\mu,0}+\delta_{\nu,0}-1}T^{ji}_{\mu\nu}(x) & (16)\end{aligned}$$

1.3 Correlators

The correlators stored into meson.dat file are computed as

$$\begin{aligned} G^{ij}_{AB}(y_0-x_0) =& \sum_{\mathbf{xy}}\langle O^{ij}_{\Gamma_A}(y)\overline{O}^{ij}_{\Gamma_B}(y)\rangle \\ =& \sum_{\mathbf{xy}}\langle \overline{q}^i(x)\Gamma_A q^j(x)\overline{q}^j(y)\gamma_0\Gamma^\dagger_B\gamma_0 q^i(y)\rangle \end{aligned} \tag{17}$$

where A, B label the gamma structures in the sink and source position respectively, x is the source position and y is the sink position. After integrating the fermionic fields, the correlator is

$$G^{ij}_{AB}(y_0-x_0) = -\sum_{\mathbf{xy}}\langle \Gamma_A S_j(x,y)\gamma_0\Gamma^\dagger_B\gamma_0 S_i(y,x)\rangle^{\text{gauge}} \tag{18}$$

When we translate operator relations, like operators imporvements or Ward identities, often we need to insert those operator in the sink position, i.e. we have to express such relations in term of bar operators. The best way of doing that is to apply the equations (13−16). For example, for the PCAC Ward identity

$$\tilde{\partial}_\mu A^{ij}_\mu(x) = 2m^{ij}P^{ij}(x), \quad m^{ij} = \frac{1}{2}(m^i_q + m^j_q) \tag{19}$$

where m^i_q is the subtracted quark mass and $\tilde{\partial}$ is the symmetric derivative

$$\tilde{\partial}_0 f(x) = \frac{1}{2a}(f(x+a) - f(x-a))) \tag{20}$$

if we use eq. (13) and (14), we get

$$\tilde{\partial}_0 \overline{A}^{ji}_0(x) - \tilde{\partial}_k \overline{A}^{ji}_k(x) = -2m^{ij}\overline{P}^{ji}(x) \tag{21}$$

and inserting this into the correlators, if projected at zero momentum, we get

$$\tilde{\partial}_{y_0} G^{ij}_{PA_0}(y_0 - x_0) = -2m^{ij}G^{ij}_{PP}(y_0 - x_0) \tag{22}$$

We see that by doing so, the Ward identity get a relative minus sign that it was not present in the original operator identity. If the correlators needed to perform the time derivative in the sink position where available, we could have used the equation

$$\tilde{\partial}_{x_0} G^{ij}_{A_0 P}(y_0 - x_0) = 2m^{ij}G^{ij}_{PP}(y_0 - x_0) \tag{23}$$

that better resembles the Ward Identity. In general this procedure has to be follow any time we want to verify or apply an operator relation.

1.4 Improvement prescriptions

Similarly as the Ward Identity, the improvements prescription are defined at the operator level, so we have to apply eqs. (13−16) to obtain the correct improvements for the correlators. At the current level, the improvements are defined as:

$$\hat{A}^{ij}_\mu(x) = A^{ij}_\mu(x) + ac_A\tilde{\partial}_\mu P^{ij} \tag{24}$$

$$\hat{V}^{ij}_\mu(x) = V^{ij}_\mu(x) + ac_V\tilde{\partial}_\nu T^{ij}_{\mu\nu}(x) \tag{25}$$

When we improve the correlators, then we have

$$
\begin{aligned}
G^{ij}_{\hat{A}_0 P}(y_0 - x_0) =& G^{ij}_{A_0 P}(y_0 - x_0) + a c_A \tilde{\partial}_{x_0} G^{ij}_{PP}(y_0 - x_0) && (26)\\
G^{ij}_{P\hat{A}_0}(y_0 - x_0) =& G^{ij}_{PA_0}(y_0 - x_0) - a c_A \tilde{\partial}_{y_0} G^{ij}_{PP}(y_0 - x_0) && (27)\\
G^{ij}_{\hat{A}_0\hat{A}_0}(y_0 - x_0) =& G^{ij}_{A_0A_0}(y_0 - x_0) + a c_A \tilde{\partial}_{x_0} G^{ij}_{PA_0}(y_0 - x_0) && (28)\\
& - a c_A \tilde{\partial}_{y_0} G^{ij}_{A_0 P}(y_0 - x_0) &&\\
G^{ij}_{\hat{V}_\mu, P}(y_0 - x_0) =& G^{ij}_{V_\mu P}(y_0 - x_0) + a c_V \tilde{\partial}_{x_0} G_{T_{\mu 0} P}(y_0 - x_0) && (29)\\
G^{ij}_{P\hat{V}_\mu}(y_0 - x_0) =& G^{ij}_{PV_\mu}(y_0 - x_0) - a c_V \tilde{\partial}_{y_0} G_{PT_{\mu 0}}(y_0 - x_0) && (30)\\
G^{ij}_{\hat{V}_\mu\hat{V}_\mu}(y_0 - x_0) =& G^{ij}_{V_\mu V_\mu}(y_0 - x_0) + a c_V \tilde{\partial}_{x_0} G_{T_{\mu 0} V_\mu}(y_0 - x_0) && (31)\\
& - a c_V \tilde{\partial}_{y_0} G^{ij}_{V_\mu T_{\mu 0}}(y_0 - x_0) &&
\end{aligned}
$$

In the equation above we just applied the improvement prescription on the correlators, hence the presence of derivative taken at the source position. In our correlators the source position is always kept fixed and we cannot compute those derivative. We can use the translation invariance

$$
\langle O(x + \Delta)\overline{O}(y + \Delta)\rangle = \langle O(x)\overline{O}(y)\rangle \tag{32}
$$

to transfer the derivate from the source to the sink position

$$
\begin{aligned}
\tilde{\partial}_{x_0}\langle O_1(x)\overline{O}_2(y)\rangle =& \frac{1}{2a}\left(\langle O_1(x_0 + a, \mathbf{x})\overline{O}_2(y)\rangle - \langle O_1(x_0 - a, \mathbf{x})\overline{O}_2(y)\rangle\right)\\
=& \frac{1}{2a}\left(\langle O_1(x)\overline{O}_2(y_0 - a, \mathbf{y})\rangle - \langle O_1(x)\overline{O}_2(y_0 + a, \mathbf{y})\rangle\right)\\
=& -\tilde{\partial}_{y_0}\langle O_1(x)\overline{O}_2(y)\rangle
\end{aligned} \tag{33}
$$

It's important to note that this invariance is exact with periodic boundary condition, while it is corrected by boundary effect with open boundary condition. These corrections decay exponentially with the distance from the boundary, hence if the souce is close to one boundary this identity might be not applicable and checks need to be done; in case the source is in the bulk of the lattice, then the boundary effects would affect the correlator when the sink is closer to the boundary, region of the lattice already affected by boundary effects.

After having transfered the derivative to the sink, when needed and applicable, we can use time reversal and charge conjugation to exchange the gamma structure and the flavour index to further simplify the improvement relations.

2 Wilson correlators

The two points functions projected to zero momentum in the Wilson data are computed using the code `meson.c` by Tomasz Korzec. We refere to his documentation for details on the algorthms used. These correlators are computing using the action

$$S = S_g[U] + \sum_{i}^{N_f} S_W^i[U, q^i, \overline{q}^i] \tag{34}$$

where $S_g[U]$ is the gluonic action and S_W is

$$S_W^f[U, q^i, \overline{q}^i] = a^4 \sum_x \overline{q}^i(x) \left[\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{a}{2}\nabla_\mu \nabla^*_\mu + m_0^i\right] q^i(x) \tag{35}$$

This action is invariant under time-reversal, parity and charge conjugation:

$$\mathcal{T} := \begin{cases} q^i(x_0, \mathbf{x}) \to \gamma_0\gamma_5 q^i(-x_0, \mathbf{x}) \\ \overline{q}^i(x_0, \mathbf{x}) \to \overline{q}^i(-x_0, \mathbf{x})\gamma_5\gamma_0 \\ U_i(x_0, \mathbf{x}) \to U_i(-x_0, \mathbf{x}) \\ U_0(x_0, \mathbf{x}) \to U_0(-x_0 - a, \mathbf{x})^\dagger \end{cases} \tag{36}$$

$$\mathcal{P} := \begin{cases} q^i(x_0, \mathbf{x}) \to \gamma_0 q^i(x_0, -\mathbf{x}) \\ \overline{q}^i(x_0, \mathbf{x}) \to \overline{q}^i(x_0, -\mathbf{x})\gamma_0 \\ U_i(x_0, \mathbf{x}) \to U_i(x_0, -\mathbf{x} - a\hat{i})^\dagger \\ U_0(x_0, \mathbf{x}) \to U_0^{(}x_0, -\mathbf{x}) \end{cases} \tag{37}$$

$$\mathcal{C} := \begin{cases} q^i(x) \to C^{-1}\overline{q}^i(x)^T \\ \overline{q}^i(x) \to -q^i(x)^T C \\ U_\mu(x) \to U_\mu(x)^* = (U_\mu(x)^\dagger)^T \end{cases} \tag{38}$$

with $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, and $C\gamma_5 C^{-1} = \gamma_5^T$

The bilinear operators (7$-$10) have an associated parity under these transformation. Correlators built with those operators inherit a parity under these trasformations that is the product of the operators parity:

$$G^{ij}_{O_1O_2}(y_0 - x_0) = \eta_{X_1}\eta_{X_2} X[G^{ij}_{O_1O_2}(y_0 - x_0)] = \eta_X X[G^{ij}_{O_1O_2}(t)] \tag{39}$$

where $X = \mathcal{T}, \mathcal{P}, \mathcal{C}$. In particular, we can relate the correlator $G^{ij}_{O_1O_2}(y_0 - x_0)$ with the correlator $G^{ij}_{O_2O_1}(y_0 - x_0)$. First we use $\mathcal{T}$, then we can exchange source and sink as

$$\begin{aligned} G^{ij}_{O_1O_2}(y_0 - x_0) \stackrel{\mathcal{T}}{=} & - \eta_{\mathcal{T}} \sum_{\mathbf{xy}} \langle \Gamma_1 S_j(\mathcal{T}x, \mathcal{T}y) \gamma_0 \Gamma_2^\dagger \gamma_0 S_i(\mathcal{T}y, \mathcal{T}x) \rangle \\ = & - \eta_{\mathcal{T}} \eta_\Gamma \sum_{\mathbf{xy}} \langle \gamma_0 \Gamma_1^\dagger \gamma_0 S_j(\mathcal{T}x, \mathcal{T}y) \Gamma_2 S_i(\mathcal{T}y, \mathcal{T}x) \rangle \\ = & - \eta_{\mathcal{T}} \eta_\Gamma \sum_{\mathbf{xy}} \langle \Gamma_2 S_i(\mathcal{T}y, \mathcal{T}x) \gamma_0 \Gamma_1^\dagger \gamma_0 S_j(\mathcal{T}x, \mathcal{T}y) \rangle \\ = & \eta_{\mathcal{T}} \eta_\Gamma G^{ji}_{O_2O_1}(y_0 - x_0) \end{aligned} \tag{40}$$

where we abuse our previous notation by defining $\eta_\Gamma = \eta_{\Gamma_1}\eta_{\Gamma_2}$ only when referring to correlators. Finally, we apply charge conjugation and obtain:

$$G^{ij}_{O_1O_2}(y_0 - x_0) = \eta_{\mathcal{T}} \eta_\Gamma \eta_{\mathcal{C}} G^{ij}_{O_2O_1}(y_0 - x_0) \tag{41}$$

For example, using this chain of identities we have that

$$G^{ij}_{A_0P}(y_0 - x_0) = G^{ij}_{PA_0}(y_0 - x_0) \tag{42}$$

since we get a change of sign only from the time-reversal of the pseudo-scalar density and for $\eta_{\gamma_5} = -1$ while all the other η-terms are equals to 1. For the complete list of such relations, see **??**

2.1 Improvement for Wilson correlators

We can us translation invariance and eq. (41) to simplify the improvements in eqs (26−31) and write

$$\begin{aligned}
G^{ij}_{\hat{A}_0 P}(y_0-x_0) =& G^{ij}_{A_0 P}(y_0-x_0) - ac_A\tilde{\partial}_{y_0} G^{ij}_{PP}(y_0-x_0) && (43)\\
G^{ij}_{P\hat{A}_0}(y_0-x_0) =& G^{ij}_{PA_0}(y_0-x_0) - ac_A\tilde{\partial}_{y_0} G^{ij}_{PP}(y_0-x_0) && (44)\\
G^{ij}_{\hat{A}_0\hat{A}_0}(y_0-x_0) =& G^{ij}_{A_0A_0}(y_0-x_0) - 2ac_A\tilde{\partial}_{y_0} G^{ij}_{A_0P}(y_0-x_0) && (45)\\
=& G^{ij}_{A_0A_0}(y_0-x_0) - 2ac_A\tilde{\partial}_{y_0} G^{ij}_{PA_0}(y_0-x_0) &&\\
G^{ij}_{\hat{V}_\mu P}(y_0-x_0) =& G^{ij}_{V_\mu P}(y_0-x_0) - ac_V\tilde{\partial}_{y_0} G^{ij}_{T_{\mu0}P}(y_0-x_0) && (46)\\
=& G^{ij}_{V_\mu P}(y_0-x_0) - ac_V\tilde{\partial}_{y_0} G^{ij}_{PT_{\mu0}}(y_0-x_0) &&\\
G^{ij}_{P\hat{V}_\mu}(y_0-x_0) =& G^{ij}_{PV_\mu}(y_0-x_0) - ac_V\tilde{\partial}_{y_0} G^{ij}_{PT_{\mu0}}(y_0-x_0) && (47)\\
G^{ij}_{\hat{V}_\mu\hat{V}_\mu}(y_0-x_0) =& G_{V_\mu V_\mu}(y_0-x_0) - 2ac_V\tilde{\partial}_{y_0} G^{ij}_{V_\mu T_{\mu0}}(y_0-x_0) && (48)
\end{aligned}$$

3 Wilson Twisted Mass correlators

The two point function in Wilson Twisted Mass (Wtm) data are computed with a mixed action setup, where in the sea there are $N_f = 2+1$ Wilson fermions and in the valece there is a wtm action with 4 flavour.

$$S_{Wtm} = \sum_i S^{(i)}_{Wtm} = a^4 \sum_i^4 \sum_x \overline{q}^i(x)(D_W + m_0^i + i\mu^i\gamma_5)q^i(x) \qquad (49)$$

The masses of the quarks $i = 1, 2$ are twisted by an angle α and the quarks $i = 3, 4$ are twisted by an angle β.

$$\tan\alpha = \frac{\mu^1}{m_0^1} = -\frac{\mu^2}{m_0^2}, \quad \tan\beta = \frac{\mu^3}{m_0^3} = -\frac{\mu^4}{m_0^4}, \quad (M^i)^2 = (m^i)^2 + (\mu^i)^2 \quad (50)$$

Notably, the twisted masses μ^1 and μ^2 have opposite sign and similary μ^3 and μ^4.

The Wtm action differs from the Wilson action only by the twisted mass term. This term changes sign under parity and time reversal trasformations, therefore breaking the $\mathcal{P}$ and $\mathcal{T}$ symmetries. However, we can apply these transformation and flip the sign of the twisted mass, so that the action stays invariant. These (spurionic) symmetries are:

$$\mathcal{P}^i \times [\mu^i \to -\mu^i] \tag{51}$$

$$\mathcal{T}^i \times [\mu^i \to -\mu^i] \tag{52}$$

Moreover, the twisted mass term breaks the γ_5-hermiticity. With the massless Wilson-Dirac operator, we have

$$\gamma_5 D_W(x,y)\gamma_5 = D_W^\dagger(y,x) \tag{53}$$

the mass term does not break this property since it just adds a scalar quatity. The twisted mass term, instead, ruins this property:

$$\gamma_5 \left[D_W(x,y) + m\,\mathbb{1} + i\mu\gamma_5\right]\gamma_5 = D_W^\dagger(y,x) + m\,\mathbb{1} + i\mu\gamma_5 \tag{54}$$

and since $\gamma_5^\dagger = \gamma_5$, we cannot restore the γ_5-hermiticy. But, if we define $\tilde{\mu} = -\mu$, we get

$$\gamma_5 \left[D_W + m\,\mathbb{1} + i\mu\gamma_5\right](x,y)\gamma_5 = \left[D_W + m\,\mathbb{1} - i\mu\gamma_5\right]^\dagger(y,x) \tag{55}$$

That is, the "γ_5-hermiticity", transforms the Dirac operator associated to a particle with mass (m,μ) into the dagger of the Dirac operator associated to a particle with mass $(m,-\mu)$. This property is used in the code that computes the 2-pt functions.

To compute the 2pt correlator in the Wtm theory, we use `tm_meson.c`, a generalization of `meson.c` by T. Korzec. The relevant difference between these two codes is only in the Dirac operator, that in tm_meson.c includes the twisted mass term.

Following Korzec documentation, to computes the correlator in eq (17), we use stochastic noise sources $\eta_{\alpha,a}(u)$ with the following properties

$$\langle \eta_{\alpha,a}(u)\rangle^n = 0 \tag{56}$$

$$\langle \eta^*_{\alpha,a}(u)\eta_{\beta,b}(v)\rangle^n = \delta_{u_0,x_0}\delta_{v_0,x_0}\delta_{\mathbf{u},\mathbf{v}}\delta_{\alpha\beta}\delta_{a,b} \tag{57}$$

Where $\langle\cdot\rangle^n$ indicate the average over the stochastic noises. The following two derived stochastic quantities may be define, this time including the twisted mass:

$$\zeta_{\alpha,a}(u) = \sum_v [D_W + m_0^i + i\mu^i\gamma_5]^{-1}_{\alpha\beta,ab}(u,v)\eta_{\beta,b}(v) \tag{58}$$

$$\xi_{\alpha,a}(u) = \sum_v [D_W + m_0^j + i\mu^j\gamma_5]^{-1}_{\alpha\beta,ab}(u,v)(\gamma_5\Gamma_A^\dagger)_{\beta\gamma,bc}\eta_{\gamma,c}(v) \tag{59}$$

Then the correlator is computed as ($\overline{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$)

$$
\begin{aligned}
G^{ij}_{AB}(y_0 - x_0; \mu^i, \mu^j) &= - \sum_{\mathbf{y}} \left\langle \left\langle \left(\overline{\Gamma}^\dagger_B \gamma_5 \xi(y) \right)^\dagger \zeta(y) \right\rangle^n \right\rangle^g \\
&= - \sum_{\mathbf{y}, vv'} \left\langle \left\langle \left(\overline{\Gamma}^\dagger_B \gamma_5 [D_W + m_0^j + i\mu^j \gamma_5]^{-1}(y,v) \gamma_5 \Gamma^\dagger_A \eta(v) \right)^\dagger \right.\right. \\
&\qquad \left.\left. \times [D_W + m_0^i + i\mu^i \gamma_5]^{-1}(y,v') \eta(v') \right\rangle^n \right\rangle^g \\
&= - \sum_{\mathbf{y}, vv'} \left\langle \left\langle \eta^*(v) \Gamma_A [D_W + m_0^j - i\mu^j \gamma_5]^{-1}(v,y) \right.\right. \\
&\qquad \left.\left. \times \overline{\Gamma}_B [D_W + m_0^i + i\mu^i \gamma_5]^{-1}(y,v') \eta(v') \right\rangle^n \right\rangle^g \\
&= - \sum_{\mathbf{y}, \mathbf{x}} \left\langle \Gamma_A S_j(x,y;-\mu^j) \overline{\Gamma}_B S_i(y,x;\mu^i) \right\rangle^g
\end{aligned}
\tag{60}
$$

where a trace over spin and color indices is understood. Since the code computes directly the quantites $\xi(u)$ and $\zeta(u)$ using the masses parameters given as input, the actual mass that enters the propagator S_j is minus the mass given as input (that is the second component in the field `mu` inside the `juobs.Corr` structure). This has some repercussion, for example in the PCVC Ward identity

$$
\partial_\mu V^{ij}_\mu = (m^i - m^j) S^{ij} + i(\mu^i - \mu^j) P^{ij} \tag{61}
$$

where the difference of twisted masses is in reality the sum of the masses given as input. In our case, the Wtm correlator have positive twisted mass input for both propagator, therefore, when we extract the term $(\mu^i - \mu^j)$ from an heavy-heavy or light-light correlators, we won't get a constant compatible with 0. This is consistent with the action simulated, and with the expectation that in the physical base the PCVC identity gets maps to the PCAC Ward identity.

3.1 Improvement for Wtm correlators

To simplify the Wilson correlators we used a combination translation invariance, $\mathcal{T}$ and $\mathcal{C}$. With Wtm correlators, though, we cannot use time reversal since it's broken by the twisted mass, but can have to use (52). Let's consider the correlator we obtain when we use as input twisted masses μ^i, μ^j,

$$G^{ij}_{O_1O_2}(y_0 - x_0; \mu^i, \mu^j) = -\sum_{\mathbf{xy}} \langle \Gamma_1 S_j(x, y; -\mu_j)\overline{\Gamma}_2 S_i(y, x; \mu_i)\rangle \tag{62}$$

when we apply

$$\mathcal{T}^i \times \mathcal{T}^j \times [\mu^i \to -\mu^i] \times [\mu^j \to -\mu^j] \tag{63}$$

we obtain:

$$\begin{aligned} G^{ij}_{O_1O_2}(y_0 - x_0; \mu^i, \mu^j) =& - \eta_{\mathcal{T}} \sum_{\mathbf{xy}} \langle \Gamma_1 S_j(\mathcal{T}x, \mathcal{T}y; -\mu^j)\overline{\Gamma}_2 S_i(\mathcal{T}y, \mathcal{T}x; \mu^i)\rangle \\ =& - \eta_{\mathcal{T}}\eta_\Gamma \sum_{\mathbf{xy}} \langle \overline{\Gamma}_1 S_j(\mathcal{T}x, \mathcal{T}y; -\mu^j)\Gamma_2 S_i(\mathcal{T}y, \mathcal{T}x; \mu^i)\rangle \\ =& - \eta_{\mathcal{T}}\eta_\Gamma \sum_{\mathbf{xy}} \langle \Gamma_2 S_i(\mathcal{T}y, \mathcal{T}x; \mu^i)\overline{\Gamma}_1 S_j(\mathcal{T}x, \mathcal{T}y; -\mu^j)\rangle \\ =& \eta_{\mathcal{T}}\eta_\Gamma G^{ji}_{O_2O_1}(y_0 - x_0; -\mu^j, -\mu^i) \end{aligned} \tag{64}$$

In this last step we obtain a correlator with opposite twisted masses. This is not a problem in our derivation, because by applying charge conjugation we now obtain:

$$\begin{aligned} G^{ij}_{O_1O_2}(y_0 - x_0; \mu^i, \mu^j) =& - \eta_{\mathcal{T}}\eta_\Gamma\eta_{\mathcal{C}} \sum_{\mathbf{xy}} \langle \Gamma_2^T S_i(\mathcal{T}x, \mathcal{T}y; \mu^i)^T \overline{\Gamma}_1^T S_j(\mathcal{T}y, \mathcal{T}x; -\mu^j)^T\rangle \\ =& - \sum_{\mathbf{xy}} \langle S_j(\mathcal{T}y, \mathcal{T}x; -\mu^j)\overline{\Gamma}_1 S_i(\mathcal{T}x, \mathcal{T}y; \mu^i)\Gamma_2\rangle \\ =& \eta_{\mathcal{T}}\eta_\Gamma\eta_{\mathcal{C}} G^{ij}_{O_2O_1}(y_0 - x_0, \mu^i, \mu^j) \end{aligned} \tag{65}$$

A Transformation property

	S^{ij}	P^{ij}	V_μ^{ij}	A_μ^{ij}	$T_{\mu\nu}^{ij}$
$\eta_\mathcal{C}$	$+$	$+$	$-$	$+$	$-$
$\eta_\mathcal{T}$	$+$	$-$	$(-1)^{\delta_{\mu 0}}$	$(-1)^{\delta_{\mu 0}-1}$	$(-1)^{\delta_{\mu 0}+\delta_{\nu 0}}$
η_Γ	$+$	$-$	$(-1)^{\delta_{\mu 0-1}}$	$(-1)^{\delta_{\mu 0}-1}$	$(-1)^{\delta_{\mu 0}+\delta_{\nu 0}-1}$
$\eta_\mathcal{C}\eta_\mathcal{T}\eta_\Gamma$	$+$	$+$	$+$	$+$	$+$

Table 1: η terms associated to charge conjugation, time reversal, and gamma structure. As shown in the last line, when exchanging the gamma structure in source and sink, the overall change of sign is always $+$

In table 1 we list all the change of sign under the various transformation we use. In particular, we highligh that, at least for the bilinear we are interested in, we have

$$G_{AB}^{ij}(y_0 - x_0) = \eta_\mathcal{T}\eta_\Gamma\eta_\mathcal{C} G_{BA}^{ij}(y_0 - x_0) = G_{BA}^{ij}(y_0 - x_0) \tag{66}$$

B Wilson Correlators

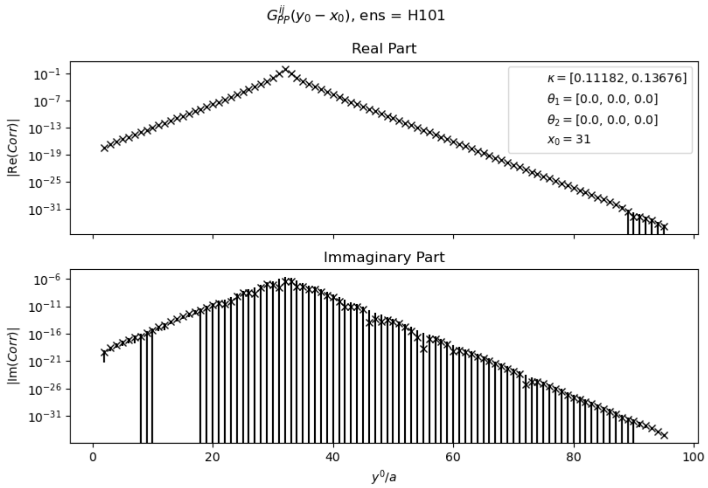
In this section we present plots and table with Wilson correlator as future reference. All correlators are heavy-light correlator of the ensemble H101 with source at $x_0 = 31a$ and $k_h = 0.111820532228709$, $k_l = 0.13675962$.

C Wtm Correlators

In this section we present plots and table with Wilson correlator as future reference. All correlators are heavy-light correlator of the ensemble H101 with source at $x_0 = 48a$ and $\mu_h = 0.2505$, $\mu_l = 0.006591$.

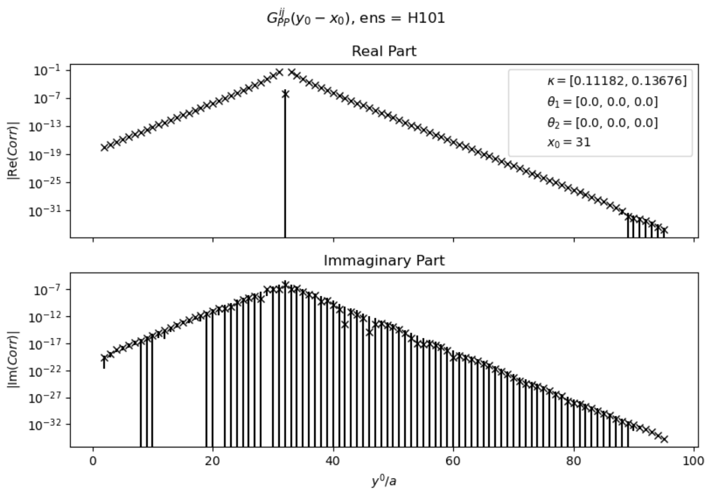
References

[1] Nikolai Husung. “Lattice artifacts of local fermion bilinears up to $O(a^2)$”. In: (Sept. 2024). arXiv: 2409.00776 [hep-lat]



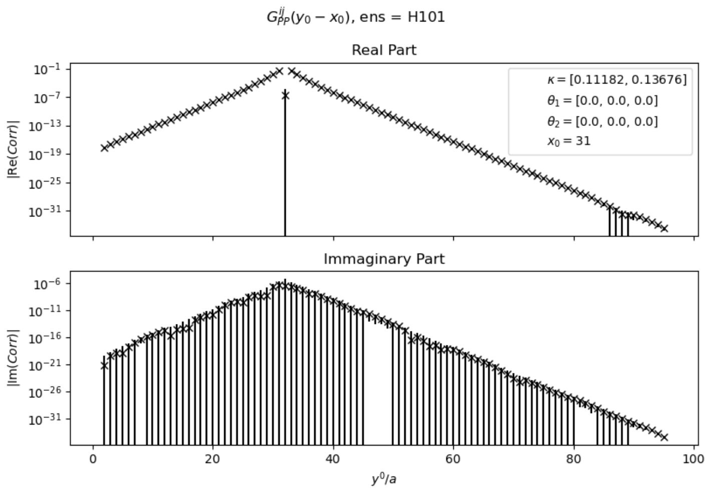
y_0/a	$\mathrm{Re}G^{hl}_{PP}$	$\mathrm{Im}G^{hl}_{PP}$
25	$2.486191e-05 \pm 3.124197e-08$	$-3.556438e-09 \pm 8.779382e-09$
26	$9.717859e-05 \pm 1.163140e-07$	$-2.096970e-09 \pm 2.423619e-08$
27	$4.122711e-04 \pm 3.909289e-07$	$-3.046994e-08 \pm 6.384367e-08$
28	$1.972254e-03 \pm 1.307918e-06$	$-1.199994e-07 \pm 1.458635e-07$
29	$1.112248e-02 \pm 4.518726e-06$	$-7.257127e-08 \pm 3.475448e-07$
30	$7.649116e-02 \pm 1.574981e-05$	$2.364976e-08 \pm 9.093381e-07$
31	$1.085041e+00 \pm 8.285549e-05$	$-3.129525e-07 \pm 1.591617e-06$
32	$7.649292e-02 \pm 1.569796e-05$	$-3.546149e-07 \pm 9.830390e-07$
33	$1.111964e-02 \pm 4.615840e-06$	$-3.902496e-08 \pm 3.954828e-07$
34	$1.971092e-03 \pm 1.389170e-06$	$4.742244e-08 \pm 1.510526e-07$
35	$4.118372e-04 \pm 4.211239e-07$	$1.190721e-08 \pm 5.591004e-08$
36	$9.699180e-05 \pm 1.268726e-07$	$1.492070e-08 \pm 2.064518e-08$

Table 2: Wilson Correlator $G^{hl}_{PP}(y_0 - x_0)$.



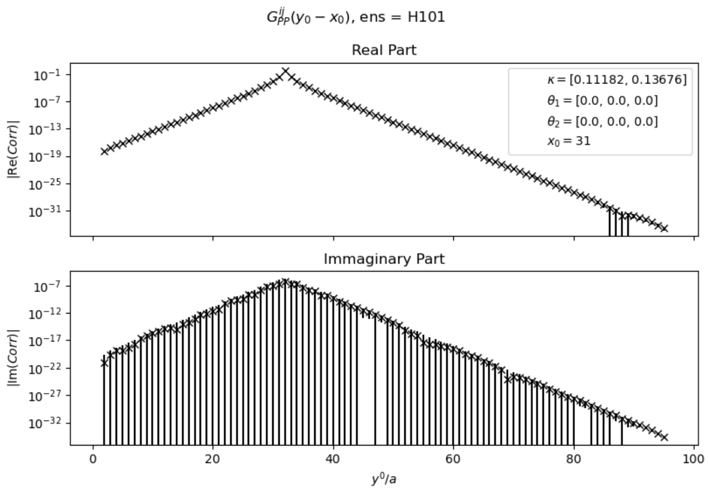
y_0/a	$\mathrm{Re}G^{hl}_{PA_0}$	$\mathrm{Im}G^{hl}_{PA_0}$
25	$-1.287254e-05 \pm 1.725698e-08$	$2.650269e-09 \pm 5.321760e-09$
26	$-4.958014e-05 \pm 6.427821e-08$	$5.613722e-09 \pm 1.402013e-08$
27	$-2.075352e-04 \pm 2.270962e-07$	$2.244840e-09 \pm 3.912408e-08$
28	$-9.893536e-04 \pm 7.747526e-07$	$1.001772e-07 \pm 9.331360e-08$
29	$-5.699571e-03 \pm 2.824753e-06$	$1.256545e-07 \pm 2.268936e-07$
30	$-4.242407e-02 \pm 9.791086e-06$	$1.129671e-07 \pm 5.384263e-07$
31	$-7.812253e-07 \pm 5.065050e-06$	$-9.292119e-07 \pm 3.345305e-06$
32	$4.241981e-02 \pm 1.022033e-05$	$1.352499e-07 \pm 6.011054e-07$
33	$5.697391e-03 \pm 2.812898e-06$	$1.633358e-07 \pm 2.429303e-07$
34	$9.886012e-04 \pm 7.870889e-07$	$3.740400e-08 \pm 9.013203e-08$
35	$2.072066e-04 \pm 2.354431e-07$	$1.267893e-08 \pm 3.360324e-08$
36	$4.946077e-05 \pm 6.932938e-08$	$7.979737e-09 \pm 1.266198e-08$

Table 3: Wilson Correlator $G^{hl}_{PA_0}(y_0 - x_0)$.



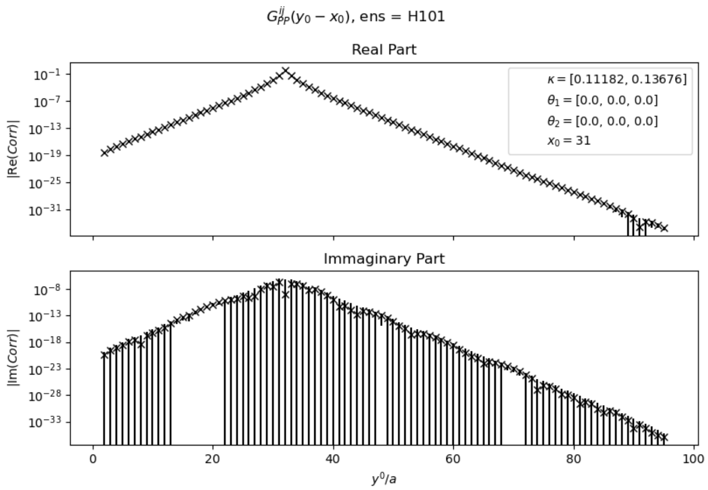
y_0/a	$\mathrm{Re}G^{hl}_{A_0P}$	$\mathrm{Im}G^{hl}_{A_0P}$
25	$-1.287844e-05 \pm 1.946489e-08$	$-2.500497e-09 \pm 8.804897e-09$
26	$-4.959043e-05 \pm 6.999619e-08$	$-5.360310e-09 \pm 2.392591e-08$
27	$-2.075036e-04 \pm 2.555751e-07$	$-3.921166e-09 \pm 6.262822e-08$
28	$-9.891883e-04 \pm 8.846066e-07$	$5.346472e-09 \pm 1.878988e-07$
29	$-5.698908e-03 \pm 3.086354e-06$	$-2.775332e-07 \pm 5.207390e-07$
30	$-4.241884e-02 \pm 1.069676e-05$	$-5.024426e-07 \pm 1.534975e-06$
31	$-3.002478e-07 \pm 5.115324e-06$	$-4.545534e-07 \pm 5.850306e-06$
32	$4.241978e-02 \pm 1.056278e-05$	$-2.293333e-07 \pm 1.483989e-06$
33	$5.696988e-03 \pm 3.061044e-06$	$1.057301e-07 \pm 4.911653e-07$
34	$9.884034e-04 \pm 8.944250e-07$	$6.444870e-08 \pm 1.747720e-07$
35	$2.071836e-04 \pm 2.559779e-07$	$1.515004e-08 \pm 6.109460e-08$
36	$4.945586e-05 \pm 7.418997e-08$	$1.412532e-08 \pm 2.153268e-08$

Table 4: Wilson Correlator $G^{hl}_{A_0P}(y_0 - x_0)$.



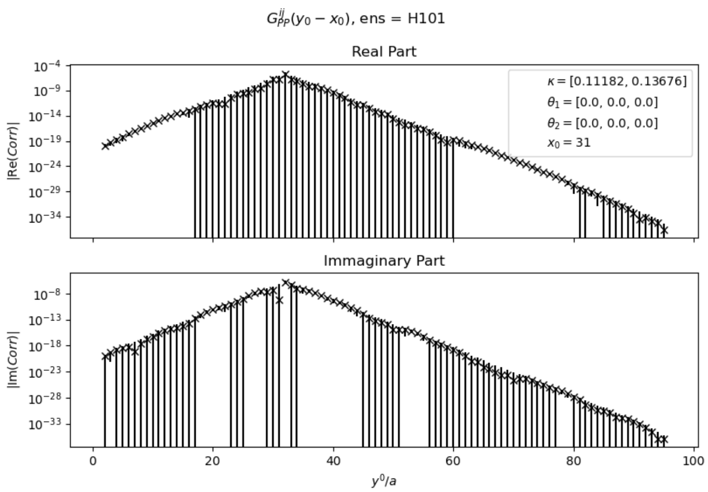
y_0/a	$\mathrm{Re}G^{hl}_{A_0A_0}$	$\mathrm{Im}G^{hl}_{A_0A_0}$
25	$6.708466e-06 \pm 1.077769e-08$	$2.202094e-09 + -5.368949e-09$
26	$2.549294e-05 \pm 3.851719e-08$	$2.433030e-09 + -1.460107e-08$
27	$1.055390e-04 \pm 1.407004e-07$	$1.462497e-08 + -4.099036e-08$
28	$5.049464e-04 \pm 5.021817e-07$	$-5.434106e-08 + -1.148338e-07$
29	$3.030030e-03 \pm 1.802804e-06$	$9.256059e-08 + -3.057771e-07$
30	$2.416162e-02 \pm 6.254325e-06$	$2.123113e-07 + -8.168979e-07$
31	$6.928420e-01 \pm 5.616387e-05$	$-5.506170e-07 + -8.911533e-07$
32	$2.415830e-02 \pm 6.385289e-06$	$1.717126e-07 + -7.978973e-07$
33	$3.028519e-03 \pm 1.758130e-06$	$2.154695e-07 + -2.766527e-07$
34	$5.044086e-04 \pm 4.940150e-07$	$3.899647e-08 + -9.982458e-08$
35	$1.053071e-04 \pm 1.426773e-07$	$1.282466e-08 + -3.648542e-08$
36	$2.540916e-05 \pm 4.164242e-08$	$1.167836e-08 + -1.303459e-08$

Table 5: Wilson Correlator $G^{hl}_{A_0P}(y_0 - x_0)$.



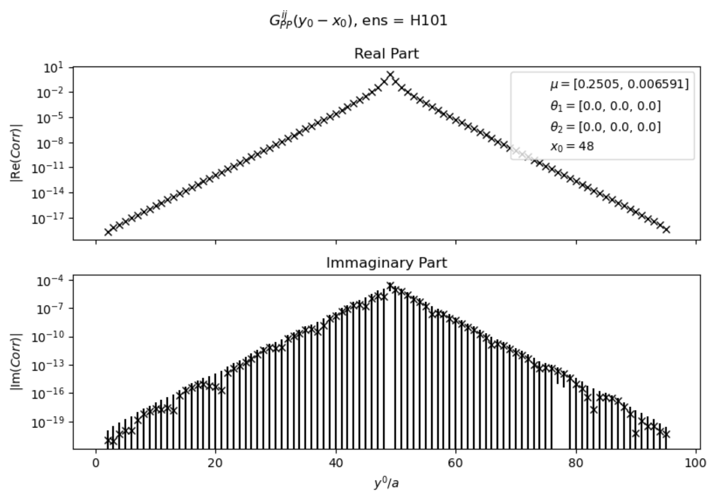
y_0/a	$\mathrm{Re}\sum_i G^{hl}_{V_iV_i}$	$\mathrm{Im}\sum_i G^{hl}_{V_iV_i}$
25	$7.530472e-06 \pm 7.946129e-09$	$2.330792e-10 \pm 4.223704e-09$
26	$3.169079e-05 \pm 2.700349e-08$	$4.855488e-10 \pm 1.185518e-08$
27	$1.484538e-04 \pm 9.964980e-08$	$-1.206275e-08 \pm 3.317670e-08$
28	$8.127320e-04 \pm 3.915240e-07$	$-4.235458e-08 \pm 9.216500e-08$
29	$5.443925e-03 \pm 1.623659e-06$	$-2.652512e-08 \pm 2.316191e-07$
30	$4.532870e-02 \pm 6.486284e-06$	$-1.937312e-07 \pm 6.157129e-07$
31	$8.820706e-01 \pm 4.946123e-05$	$-1.003070e-09 \pm 7.347670e-07$
32	$4.532746e-02 \pm 6.444943e-06$	$1.062579e-07 \pm 6.422132e-07$
33	$5.442420e-03 \pm 1.588118e-06$	$1.145161e-07 \pm 2.386517e-07$
34	$8.122190e-04 \pm 4.211390e-07$	$4.854620e-08 \pm 8.106919e-08$
35	$1.482929e-04 \pm 1.088383e-07$	$5.804953e-09 \pm 2.839190e-08$
36	$3.164126e-05 \pm 2.866634e-08$	$9.643884e-09 \pm 1.010901e-08$

Table 6: Wilson Correlator $\sum_i G^{hl}_{V_iV_i}(y_0 - x_0)$. In bold, the correlator at source



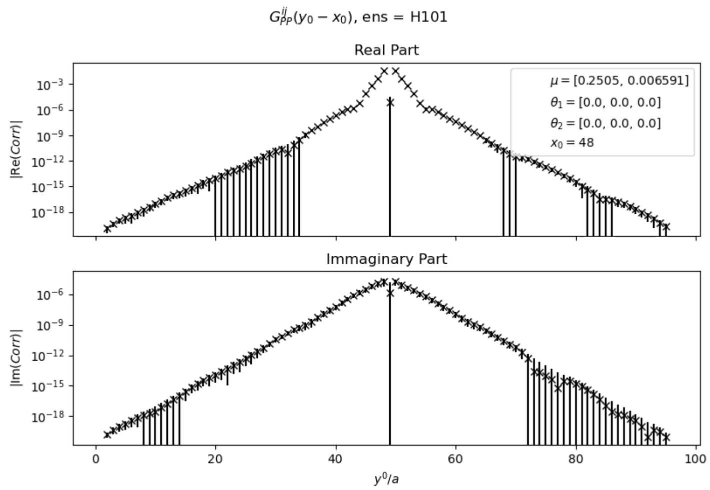
y_0/a	$\mathrm{Re}\sum_i G^{hl}_{PV_i}$	$\mathrm{Im}\sum_i G^{hl}_{PV_i}$
25	$-4.043898e-10 \pm 3.155288e-09$	$4.394422e-09 \pm 2.831789e-09$
26	$-2.079001e-09 \pm 9.544043e-09$	$1.293778e-08 \pm 8.099380e-09$
27	$2.553873e-09 \pm 2.711571e-08$	$2.990256e-08 \pm 2.263081e-08$
28	$2.522827e-08 \pm 8.473741e-08$	$1.800841e-08 \pm 6.488688e-08$
29	$1.742748e-07 \pm 2.472790e-07$	$5.107669e-08 \pm 1.940098e-07$
30	$1.508414e-07 \pm 7.192299e-07$	$6.206938e-10 \pm 6.328354e-07$
31	$-1.438017e-06 \pm 3.019276e-06$	$1.667595e-06 \pm 1.091171e-06$
32	$-1.434966e-07 \pm 7.380380e-07$	$4.146468e-07 \pm 6.730423e-07$
33	$-7.322061e-08 \pm 2.631148e-07$	$8.645650e-08 \pm 1.968085e-07$
34	$-2.454325e-08 \pm 1.011650e-07$	$7.408829e-08 \pm 6.396367e-08$
35	$7.337338e-09 \pm 3.495451e-08$	$3.694000e-08 \pm 2.198272e-08$
36	$7.156181e-09 \pm 1.165281e-08$	$1.341531e-08 \pm 7.310301e-09$

Table 7: Wilson Correlator $\sum_i G^{hl}_{PV_i}(y_0 - x_0)$.



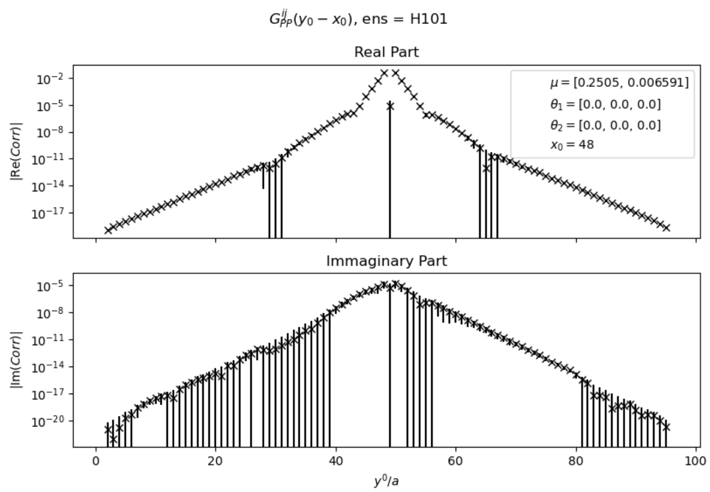
y_0/a	$\mathrm{Re}G^{hl}_{PP}$	$\mathrm{Im}G^{hl}_{PP}$
43	$1.189411e-03 \pm 1.231820e-06$	$2.475078e-07 \pm 5.420663e-07$
44	$3.400552e-03 \pm 2.759237e-06$	$1.747158e-07 \pm 1.164030e-06$
45	$1.080115e-02 \pm 6.408774e-06$	$1.073637e-06 \pm 2.445479e-06$
46	$3.999353e-02 \pm 1.357811e-05$	$2.240500e-06 \pm 5.111593e-06$
47	$1.790100e-01 \pm 3.105115e-05$	$1.709271e-06 \pm 1.004489e-05$
48	$1.464734e+00 \pm 1.405126e-04$	$-2.627281e-05 \pm 2.048971e-05$
49	$1.790302e-01 \pm 3.020278e-05$	$8.705821e-06 \pm 1.168832e-05$
50	$4.001016e-02 \pm 1.338394e-05$	$6.011066e-06 \pm 6.187975e-06$
51	$1.080950e-02 \pm 6.318871e-06$	$2.437541e-06 \pm 2.863384e-06$
52	$3.404624e-03 \pm 2.791766e-06$	$9.444601e-07 \pm 1.324158e-06$
53	$1.191431e-03 \pm 1.216378e-06$	$4.743012e-07 \pm 6.077831e-07$

Table 8: Wtm Correlator $G^{hl}_{PP}(y_0 - x_0)$.



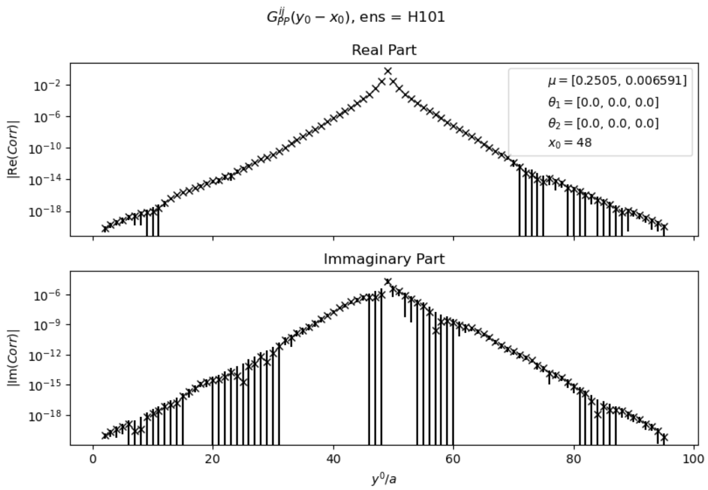
y_0/a	$\mathrm{Re}G^{hl}_{A_0P}$	$\mathrm{Im}G^{hl}_{A_0P}$
43	$-6.824477e-06 \pm 1.045232e-06$	$1.513633e-06 \pm 7.699842e-07$
44	$-8.115043e-05 \pm 2.411764e-06$	$3.153351e-06 \pm 1.654132e-06$
45	$-6.381409e-04 \pm 5.660985e-06$	$6.345191e-06 \pm 3.353235e-06$
46	$-4.859477e-03 \pm 1.313511e-05$	$1.138424e-05 \pm 6.637218e-06$
47	$-4.294598e-02 \pm 3.121562e-05$	$1.749129e-05 \pm 1.240729e-05$
48	$-8.442875e-06 \pm 2.140130e-05$	$1.406853e-06 \pm 1.365984e-05$
49	$4.294786e-02 \pm 3.180062e-05$	$-1.699791e-05 \pm 1.056503e-05$
50	$4.859853e-03 \pm 1.364382e-05$	$-8.390201e-06 \pm 5.301045e-06$
51	$6.375116e-04 \pm 5.340981e-06$	$-4.593899e-06 \pm 2.675382e-06$
52	$8.073885e-05 \pm 2.237467e-06$	$-2.408645e-06 \pm 1.322299e-06$
53	$6.721926e-06 \pm 9.608762e-07$	$-1.166467e-06 \pm 6.351978e-07$

Table 9: Wtm Correlator $G^{hl}_{A_0P}(y_0 - x_0)$.



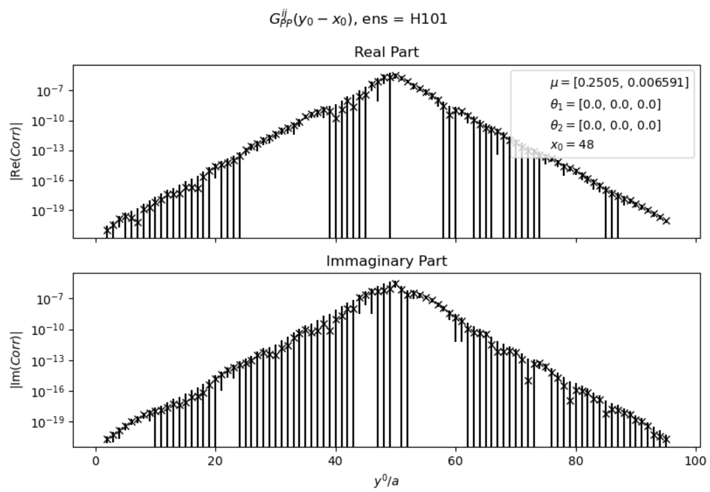
y_0/a	$\mathrm{Re}G^{hl}_{PA_0}$	$\mathrm{Im}G^{hl}_{PA_0}$
43	$-7.138467e-06 \pm 1.019243e-06$	$9.882506e-07 \pm 5.228413e-07$
44	$-8.190232e-05 \pm 2.396660e-06$	$1.949448e-06 \pm 1.085912e-06$
45	$-6.398322e-04 \pm 5.662586e-06$	$3.336730e-06 \pm 2.257812e-06$
46	$-4.862598e-03 \pm 1.297125e-05$	$5.778033e-06 \pm 4.618691e-06$
47	$-4.293581e-02 \pm 3.378026e-05$	$1.290550e-05 \pm 9.094534e-06$
48	$8.466922e-06 \pm 2.032224e-05$	$5.038022e-06 \pm 1.183041e-05$
49	$4.295613e-02 \pm 3.144893e-05$	$-1.564380e-05 \pm 1.149279e-05$
50	$4.869313e-03 \pm 1.205280e-05$	$-7.739129e-06 \pm 5.970358e-06$
51	$6.421065e-04 \pm 5.013272e-06$	$-2.330490e-06 \pm 2.923747e-06$
52	$8.330745e-05 \pm 2.139743e-06$	$-6.950232e-07 \pm 1.322021e-06$
53	$7.957601e-06 \pm 8.835687e-07$	$-8.163809e-08 \pm 6.107786e-07$

Table 10: Wtm Correlator $G^{hl}_{PA_0}(y_0 - x_0)$.



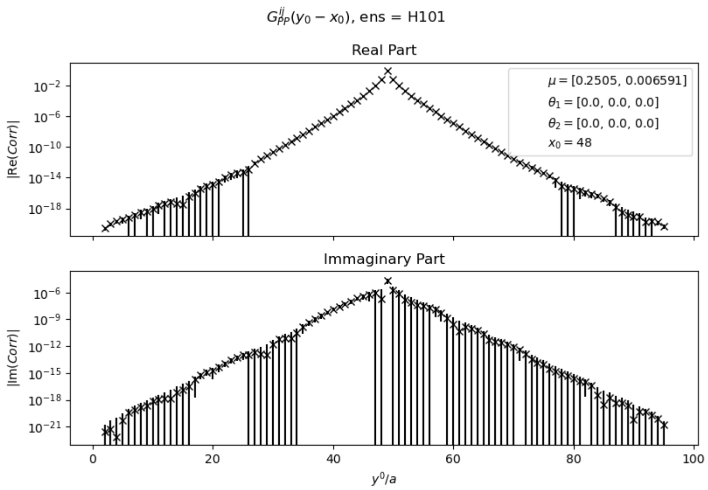
y_0/a	$\mathrm{Re}G^{hl}_{A_0A_0}$	$\mathrm{Im}G^{hl}_{A_0A_0}$
43	$4.737117e-05 \pm 2.000177e-07$	$-3.047203e-07 \pm 1.424749e-07$
44	$1.643441e-04 \pm 4.477511e-07$	$-5.389713e-07 \pm 3.039705e-07$
45	$6.799198e-04 \pm 9.969896e-07$	$-5.016022e-07 \pm 6.450735e-07$
46	$3.730409e-03 \pm 2.227283e-06$	$-5.740298e-07 \pm 1.394204e-06$
47	$2.630513e-02 \pm 4.323109e-06$	$9.447859e-07 \pm 3.086243e-06$
48	$7.023814e-01 \pm 6.778031e-05$	$-2.113072e-05 \pm 1.094085e-05$
49	$2.630696e-02 \pm 4.461104e-06$	$3.587924e-06 \pm 3.078152e-06$
50	$3.729709e-03 \pm 2.144127e-06$	$2.032911e-06 \pm 1.442532e-06$
51	$6.796088e-04 \pm 9.478821e-07$	$7.228614e-07 \pm 7.174325e-07$
52	$1.640044e-04 \pm 4.225101e-07$	$3.313281e-07 \pm 3.297873e-07$
53	$4.715876e-05 \pm 1.891883e-07$	$1.508204e-07 \pm 1.672118e-07$

Table 11: Wtm Correlator $G^{hl}_{A_0A_0}(y_0 - x_0)$.



y_0/a	$\mathrm{Re}\sum_i G^{hl}_{PV_i}$	$\mathrm{Im}\sum_i G^{hl}_{PV_i}$
43	$2.838222e-08 \pm 8.941215e-08$	$1.165926e-07 \pm 1.133105e-07$
44	$-4.295063e-08 \pm 1.868798e-07$	$2.431866e-07 \pm 2.200818e-07$
45	$-4.706102e-07 \pm 3.874014e-07$	$4.643974e-07 \pm 4.614302e-07$
46	$-8.553009e-07 \pm 8.476058e-07$	$4.924849e-07 \pm 1.009735e-06$
47	$-2.415524e-06 \pm 1.918889e-06$	$-6.053224e-07 \pm 2.207011e-06$
48	$-2.356286e-06 \pm 4.538811e-06$	$8.573523e-07 \pm 3.737835e-06$
49	$-3.165025e-06 \pm 1.770456e-06$	$2.967837e-06 \pm 1.860003e-06$
50	$-1.748388e-06 \pm 8.231411e-07$	$7.592610e-07 \pm 8.931665e-07$
51	$-9.160106e-07 \pm 3.866593e-07$	$-2.256812e-07 \pm 4.488743e-07$
52	$-3.205484e-07 \pm 1.836741e-07$	$-3.329482e-07 \pm 2.451831e-07$
53	$-1.701005e-07 \pm 8.618370e-08$	$-2.297141e-07 \pm 1.258355e-07$

Table 12: Wtm Correlator $\sum_i G^{hl}_{PV_i}(y_0 - x_0)$.



y_0/a	$\mathrm{Re}\sum_i G^{hl}_{V_iV_i}$	$\mathrm{Im}\sum_i G^{hl}_{V_iV_i}$
43	$1.369432e-04 \pm 3.002924e-07$	$-2.209504e-07 \pm 1.035094e-07$
44	$5.272253e-04 \pm 7.321605e-07$	$-3.917349e-07 \pm 2.364048e-07$
45	$2.319186e-03 \pm 1.688252e-06$	$-6.039082e-07 \pm 5.052445e-07$
46	$1.221175e-02 \pm 3.438041e-06$	$-8.611630e-07 \pm 1.083811e-06$
47	$7.750563e-02 \pm 1.542489e-05$	$-1.974446e-07 \pm 2.126789e-06$
48	$1.060008e+00 \pm 8.244740e-05$	$-2.033721e-05 \pm 1.066998e-05$
49	$7.750281e-02 \pm 9.230063e-06$	$1.766988e-06 \pm 2.375912e-06$
50	$1.221046e-02 \pm 3.366478e-06$	$7.530092e-07 \pm 1.026214e-06$
51	$2.318048e-03 \pm 1.610928e-06$	$1.612890e-07 \pm 4.835651e-07$
52	$5.266720e-04 \pm 6.837530e-07$	$8.063977e-08 \pm 2.255573e-07$
53	$1.367048e-04 \pm 2.986236e-07$	$-4.367116e-08 \pm 1.021484e-07$

Table 13: Wtm Correlator $\sum_i G^{hl}_{V_iV_i}(y_0 - x_0)$.